Holographic Effective Actions from Black Holes

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Using the Wald's relation between the Noether charge of diffeomorphisms and the entropy for a generic spacetime possessing a bifurcation surface, we introduce a method to obtain a family of higher order derivatives effective actions from the entropy of black holes. We consider the entropy as the starting point and we analyze the procedure of derivation of the action functional. We specialize to a particular class of theories which simplifies the calculations, $f(R)$ theories. We apply the procedure to loop quantum gravity and to a general class of log-corrected entropy formulas.

I. INTRODUCTION

There is a deep connection between thermodynamics and general relativity. Since the remarkable discovery that black holes posses entropy, this connection between Einstein general relativity and thermodynamics has been slowly unveiled by the work of few researchers [1][2][3][5]. A relation between the second law of black holes and the diffeomorphisms charge for a generic diff-invariant Lagrangian was found by Wald in [1]. Assuming as a starting point holography, Jacobson [5] found that the Einstein equations have a thermodynamical origin. Recently more work has been put forth by Padmanabhan [2] and during the drawing up of this draft by Verlinde [6], followed by Smolin [7]. In this paper we study a formalism to read back from a generic black hole entropy an effective (higher order) Lagrangian theory [8][9][10], as emphasized in the following cartoon,
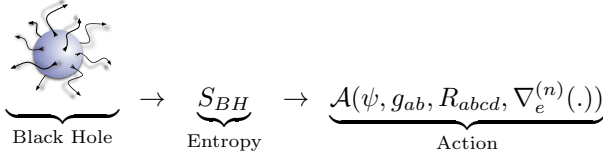
where the derivative $\nabla_e^{(n)}(.)$ in the argument acts on any field ψ or the Riemann tensor in all possible ways. We are applying *contrariwise* the procedure in literature: we start from the entropy and we obtain the effective bulk action. However this procedure is only applicable if the theory we are considering can admits black holes. This relation is supported by Padmanabhan papers [3] where it is explicitly shown a relation between the bulk and the surface terms of the Lagrangian,

$$\sqrt{-g}L_{\text{sur}} = -\partial_a\left(g_{ij}\frac{\partial\sqrt{-g}L_{\text{bulk}}}{\partial(\partial_a g_{ij})}\right)$$

and that the surface term in the action has a direct thermodynamical meaning as the horizon black hole entropy. Surprisingly, this relation is true also for a U(1) gauge theory, the bulk Maxwell theory can be derived from a boundary term [4]. However the surface term has no natural interpretation since the original bulk action does not have second derivatives.

The structure of the paper is as follows. In section 2 we review the relation between Wald's entropy and a generic diff-invariant Lagrangian and we recall the general idea and derive the relation between modified black hole entropies and the $f(R)$ actions and derive the Einstein-Hilbert lagrangian. In section 3 we introduce a relation between the black hole event horizon area and the scalar curvature based on scaling grounds evaluated on the bifurcation surface; such relation is suggested by loop quantum black holes [11]. We use such scaling argument to find a family of possible Lagrangians which are Planck-scale log-corrected higher-order $f_\beta(R)$ theories. The free parameter β represents the scaling exponent between the curvature and the area of the black hole in our ansatz. In section 4 conclusions follow.

II. BLACK HOLE ENTROPY AND $f(R)$ THEORIES

Let consider a general Lagrangian of the following form,

$$L = L(\psi, \nabla_a\psi, g_{ab}, R_{abcd}, \nabla_{(e_1}\dots\nabla_{e_n)}R_{abcd}), \qquad (1)$$

which in principle may contain matter fields, the Riemann tensor and the symmetric derivatives of R_{abcd} from one to order n. In what follows we represent all possible fields in the Lagrangian functional with χ. A generic variation of (1) is of the form:

$$\delta(\sqrt{-g}L) = \sqrt{-g}E \cdot \delta\chi + \sqrt{-g}\nabla_a\theta^a(\delta\chi) \qquad (2)$$

where "$\cdot$" represents the tensorial contraction, $E = 0$ are the equations of motion, θ^a is the associated Noether current which is conserved *on shell* (as can be seen from (2)). In the case of diffeomorphisms the variation of fields is the Lie derivative, $\mathscr{L}_\xi(\cdot)$. In this case we can replace the Noether current with

$$J = \theta(\mathscr{L}_\xi) - \iota_\xi L \qquad (3)$$

which is still conserved *on shell*. A direct calculation shows that

$$\delta L = Y^{abcd}(-2\nabla_a(\nabla_c\delta g_{bd}))\epsilon + \cdots,$$

where $Y^{abcd} = \partial L / \partial R_{abcd}$ and ϵ is the volume form. Here and in the following the ellipses hide terms which do not contribute. From this expression we can calculate the associated Noether charge:

$$Q(\xi) = -Y^{abcd} \nabla_c \xi_d \hat{\epsilon}_{ab} + \cdots .$$

We can now replace Q with its density, $\nabla_c \xi_d$ with $\hat{\epsilon}_{ab}$ and $\epsilon_{ab} = \epsilon \hat{\epsilon}_{ab}$ where the hat is to specify it is the binormal to the surface. Then, the identity proved by Wald between this Noether charge and the entropy is:

$$S_{BH} = -2\pi \oint Q(\xi) = -2\pi \oint Y^{abcd} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \epsilon . \tag{4}$$

All this machinery becomes much more interesting when we restrict to the case in which ξ is a Killing vector for the fields (such that $\mathscr{L}_\xi \chi = 0$) and we choose a hypersurface which is a bifurcation surface, where the timelike Killing vector field vanishes. For static/stationary black holes event horizon this is the case. Using such a procedure Wald proved the first law of black holes mechanics while here we focus on the relation between the entropy and the Noether charge of diffeomorphisms for a diff-invariant Lagrangian. A generalization of (4) for a generic diff-invariant Lagrangian, in units of surface gravity, is:

$$S_{BH} = -2\pi \int_{\partial\mathcal{H}} d^2x \sqrt{h} \sum_{m=0}^{n} \nabla_{(e_1} ... \nabla_{e_m)} Z^{e_1...e_m;abcd} \hat{\epsilon}_{ab} \, \hat{\epsilon}_{cd},$$

$$Z^{e_1...e_n;abcd} := \frac{\partial L}{\partial \nabla_{(e_1} ... \nabla_{e_n)} R_{abcd}} . \tag{5}$$

For the details of the derivation we address to the original papers by Wald et al. [1] and [9]. The metric h_{ab} is the one induced on the horizon $\mathcal{H}$, ϵ_{ab} is the bi-normal to $\partial\mathcal{H}$. Let us now consider a spherically symmetric metric of the form ,

$$ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + q(r)d\Omega^{(2)}, \tag{6}$$

where we suppose it has a bifurcation surface. Then we can write the entropy in the generic form ,

$$S_{BH} = \frac{A}{4} \times s(A), \tag{7}$$

where $s(A)$ is a general function of the event horizon area. In order to simplify the discussion and show clearly the main step of the procedure, we consider the Lagrangian to be an $f(R)$ theory[12][13],

$$\mathcal{L} = \frac{1}{16\pi G_N} f(R), \tag{8}$$

where R is the trace of the Ricci tensor. If we use the black hole entropy formula (5) found using Wald's theorem, we find:

$$S_{BH} = -2\pi \int_{\mathcal{H}} \frac{\partial L}{\partial R_{abcd}} \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \, q(r) \, d\Omega^{(2)}$$

$$= -2\pi \int_{\mathcal{H}} f'(R) \frac{1}{2} (g^{ac} g^{bd} - g^{ad} g^{bc}) \hat{\epsilon}_{ab} \hat{\epsilon}_{cd} \, q(r) \, d\Omega^{(2)}$$

$$= -\frac{1}{4} \int_{t_0, r_H} (g^{tt} g^{rr} - g^{tr} g^{rt}) f'(R) \, q(r) \, d\Omega^{(2)}, \tag{9}$$

where the prime means derivation with respect to the Ricci scalar. For a spherically symmetric black hole while of the form given in (6) we have $g^{tr} = g^{rt} = 0$ and $g^{tt} g^{rr} = \text{const.} = -k$, the surface gravity and we can identify (7) with (8) and then obtain the Lagrangian

$$f(R) = \frac{1}{k} \int s(A(R)) dR . \tag{10}$$

If $s(A) \neq 1$, the missing step in the procedure is the relation between the Ricci scalar evaluated on the horizon and the event horizon area. Given such a relation we can obtain the first derivative of the $f(R)$ Lagrangian and we get an $f(R)$-theory modulo a cosmological constant term. Let us notice that there is a nontrivial step in the procedure: the Lagrangian is a functional, while the Area-Entropy relation is a relationship between *functions*. At a first sight (10) could be simply stated as wrong because of this inconsistency. However, the curvature is constant over the bifurcation surface, thus the relation collapses on an ordinary integral and *not* a functional integration, as it should be if the entropy (9) would not have been evaluated on the bifurcation surface. Moreover, it might seem that the argument is valid *only* for spherically symmetric spacetimes. However, a careful thought will convince that this relation between (9) and (8) is independent from the particular spacetime. The use of a spherically symmetric spacetimes is thus just a mere simplification in the derivation. The true missing step in this procedure is the relation $A(R)$ between the area and the curvature *on* the bifurcation surface. However we want to keep the discussion general and assume this relation as given. We would like to discuss here the case of general relativity. The entropy for a Schwartzschild black hole in general relativity is, in units where the Boltzmann constant and the Planck length are both equal to one, $S_{BH} = A/4$, as it is well known. The correction function is $s(A) = 1$ and integrating (10) for $k = 1$ (independently from the relation between the area and the Ricci curvature) we find

$$f(R) = R + \text{const.}, \tag{11}$$

which is the Einstein-Hilbert Lagrangian with cosmological constant. It should be stressed that in this case the procedure does not need any further information but the area-entropy relationship. Moreover it is important that the Area-Entropy relationship is of the form (7). The cosmological constant is an integration constant of the procedure. In the more general case of a quantum gravity corrected entropy, the requirement for $s(A) \equiv s(A, l_p)$ is:

$$\lim_{l_p \to 0} s(A, l_p) = 1 \tag{12}$$

in order to have the correct classical limit.

Let us briefly discuss the physical input in this set of assumptions and how the action can be considered effective, that means that while it must considered phenomenological, it is neglecting something. In order to understand
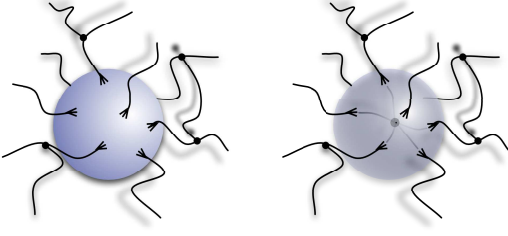
FIG. 1: Black hole spin-network. The picture on the left represents the spin-network graph outside the black hole and the punctures on the event horizon. The pictures on the right represents also the spin-network inside the black hole horizon. In particular the space inside the horizon is characterized by a single node and the entropy by the intertuner space dimensionality.

this, let us recall the main steps of Jacobson's derivation of the Einstein equations and the extension to $f(R)$-gravity theories. We follow step by step [13]. Consider a free falling observer (particle) in space time located at a point p. To the point p we can attach a local *quasi*-flat set of coordinates using the equivalence principle. Locally, due to the approximate Poincaré symmetry, there is a Rindler horizons $\mathcal{H}$ which is also the causal barrier for the particle. Due to the passage of the particle, to the Rindler horizon $\mathcal{H}$ there is also an associated time-like Killing vector past-directed and a heat flow. The variation of the heat is given by $\delta Q = - \int_{\mathcal{H}} ksT_{ab}K^a K^b dsdA$, where s is the affine parameter associated to the trajectory of the particle, K^a the generators of the Rindler horizon, dA is the element of area associated to the horizon and k is the surface gravity. Also, there is a temperature associated to the horizon which is the Unruh temperature, and thus an entropy. Jacobson now assumes that there a proportionality relation between the element of area and the element of entropy, through a constant, η, which is unkown, $\delta S = \eta \delta A$. Using the Raychaudhury equation, then the element of area δA is given by $\delta A = \int_{\mathcal{H}} \theta(s)dsdA$, where $\theta(s)$ is the geodesic deviation (note: the shear is assumed *zero*). Now let make an ansatz[9] for the parameter η:$\eta \equiv \eta(g, R_{abcd}, \nabla R_{abcd})$. Remarkably, using the Wald-Iyer procedure [13], it can be now shown that for $f(R)$ gravity we have:

$$S = \frac{A}{4G_{eff}}$$
$$\frac{1}{8\pi G_e ff} = E^{abcd}\hat{\epsilon}_{ab}\hat{\epsilon}_{cd} = f'(R) = \frac{\eta}{2\pi} \qquad (13)$$

and then can be integrated exactly as Jacobson did for the Einstein-Hilbert case. Thus, the physical input is the following: gravity is associated to the a *shear-free* (here is the effectiveness) fluid and the (modified) local degrees of freedom associated to the gravitons are encoded into the effective Newton constant G_{eff}. In our recipy this is the meaning to give to the scaling relation between the area of the horizon and the Ricci scalar.

III. LQG BLACK HOLE ENTROPY

String theory and Loop Quantum Gravity are two approaches to quantum gravity able to obtain the Bekenstein-Hawking entropy of a black hole [14][15]. It is a generic feature of black hole entropy derived from quantum gravity theories to contain a logarithmic correction to the Bekenstein-Hawking entropy. LQG is one of these theories and in the following we focus on this theory. It is interesting that in some cases [18] the effective action calculated from loop quantum cosmology has a logarithmic correction to the Einstein-Hilbert action. In LQG the black hole entropy is given by the logarithm of the number of microstates compatible with a fixed event horizon area (macrostate). Here we rederive the LQG entropy result in a *a posteriori* simplified way. In LQG the number of microstates on the horizon of a black hole is given by the dimension of the intertwiner space hidden beyond the event horizon. According to the area operator of LQG, the area spectrum of the horizon is given by the punctures crossing the horizon in a j_p representation of $SU(2)$, and reads

$$A = 8\pi l_P^2 \gamma \sum_{j_p}^{n} \sqrt{j_p(j_p + 1)}, \qquad (14)$$

while the number of microstate is instead given by the tensor product $N = \dim \mathrm{Inv}[\otimes_p \mathcal{H}_{j_p}]$, and can be defined using an integral formula,

$$N = \frac{2}{\pi} \int_0^\pi d\theta \sin^2(\theta) \prod_p^n \left(\frac{\sin((2j_p + 1)\theta)}{\sin(\theta)} \right). \qquad (15)$$

The entropy is defined, given the number of macrostates N, $S_{BH} = \log N$. However, in order to obtain the entropy we need to solve (14) and (15) together. We should invert (14) for n, insert the result in (15) and take the logarithm. However it is clear that such relation cannot be solved analytically. We can simplify the problem by taking all the representations across the event horizon to be equal. In this case it is very simple to invert (14) for n and obtain $n = A/[8\pi l_P^2 \gamma \sqrt{j(j+1)}]$. The dimension of the Hilbert space for equal representations $j_1 = j_2 = \ldots j_n = j$ simplifies to,

$$N = \frac{2}{\pi} \int_0^\pi d\theta \sin^2(\theta) \left[\sin((2j + 1)\theta)/\sin(\theta) \right]^n .$$

It is empirically known that the $j = 1/2$ case dominates the entropy. In this case we can explicitly evaluate the integral (15), obtaining the result

$$S_{BH} = \log \left[\frac{(-2)^n \sqrt{\pi} \cos(\frac{n\pi}{2})}{\Gamma(\frac{1-n}{2}) \Gamma(\frac{n+4}{2})} \right], \qquad (16)$$

where n can be expressed in terms of the area eigenvalue using (14) with $j = 1/2$. This analytic formula will turn out to be useful. Using the property $\Gamma(z)\Gamma(1 - z) =$

$\pi/\sin(\pi z)$ and the Stirling's approximation $\log(\Gamma(z)) \approx 1/2 \log(2\pi) + (z - 1/2) \log(z) - z$, we can obtain the black hole entropy for a large number of punctures

$$S_{BH} = n \log(2) - \frac{3}{2} \log(n) + \frac{1}{2} \log(8/\pi). \qquad (17)$$

When we replace the number of punctures with the area spectrum (for $j = 1/2$) we obtain

$$S_{BH} = \frac{A}{4} - \frac{3}{2} \log(A) + \frac{1}{2} \log\left(\frac{8}{\pi}\right) + \frac{3}{2} \log(4 \log 2). \quad (18)$$

This has the form of a logarithmically corrected Bekenstein-Hawking entropy, as anticipated. Given this Area-Entropy relation we can try to find an $f(R)$ functional that reproduces the given entropy as done previously in this paper. First, we consider this procedure in a semiclassical limit, thus for a large number of punctures in (18). Then, the missing crucial ingredient is the relation between the trace of the Ricci tensor (evaluated on the horizon) and the event horizon area. A correct ansatz should be a generic relation between the horizon area and the Ricci scalar, contractions of Ricci tensor and contractions of Riemann tensor. However, in order to keep the discussion simple we choose the simplest ansatz, a scaling law relation between the area of the horizon and the Ricci scalar, of the form

$$A(R) = \frac{c}{a_o^{\beta-1} R^\beta}, \qquad (19)$$

where $a_o := l_P^2$, c is a constant and $\beta > 0$ (as we will see later). The ansatz (19) is an exact relation for the black hole solution obtained in [11] with $\beta = 1$ and we believe to be a natural choice based on dimensional arguments. Moreover it allows us to carry out the procedure until the end and find an action depending the parameters β, a_0, c and α. Given this simplification, the entropy correction $s(A, l_p)$ obtained from (18) using (19) is

$$s(A(R)) = 1 - \frac{6 \, a_o \log(A/a_o)}{A}$$

$$= 1 - \frac{6 \, a_o^\beta R^\beta \log(c/R^\beta a_o^\beta)}{c}. \qquad (20)$$

Inserting (20) into (10), we obtain the following Lagrangian

$$f(R) = R + \frac{6 \, a_o^\beta}{c(\beta+1)^2} R^{\beta+1} \left[(\beta+1) \log\left(\frac{R^\beta a_o^\beta}{c}\right) - \beta \right].$$

To obtain this result we assumed that the Ricci scalar is not zero on the horizon. While this assumption is needed in this case, it was not necessary in the case of the general relativity (where $R = 0$ on the Schwarzschild horizon). The correct semiclassical limit was already guaranteed by (12) applied to the LQG case.

IV. GENERIC log-CORRECTED ENTROPY

The analysis applied to LQG is of general character and can applied to any black hole entropy with a logarithmic correction [16] (see [17] for a recent discussion in LQG). We parametrize the entropy with two parameters $\epsilon > 0$, consistent with the Bekenstein bound, and the Planck length l_P:

$$S_{BH} = \frac{A}{4a_o} - \epsilon \log\left(\frac{A}{a_o}\right) - \alpha. \qquad (21)$$

We also restrict attention to $f(R)$-theories and we suppose the same proportionality relation between the event horizon area and Ricci scalar (19). The reduced function is

$$s(A) = 1 - \frac{4a_o}{A} \left(\epsilon \log\left(\frac{A}{a_o}\right) + \alpha \right). \qquad (22)$$

The effective action for the general entropy form is, for $\beta \neq -1$,

$$f_\beta(R) = R - \frac{4 \, a_o^\beta(\alpha\beta + \alpha + \beta\epsilon)}{c(\beta+1)^2} R^{\beta+1} + \frac{4 \, a_o^\beta \epsilon}{c(\beta+1)} R^{\beta+1} \log\left(\frac{a_o^\beta R^\beta}{c}\right). \qquad (23)$$

where we omitted a cosmological constant term.

Let put $q = 4a_0^\beta/c(\beta+1)$ for simplicity. Then equation (23) becomes:

$$f_\beta(R) = R \left\{ 1 - qR^\beta \left[\frac{\alpha\beta + \alpha + \beta\epsilon}{\beta+1} - \epsilon \log\left(\frac{a_o^\beta R^\beta}{c}\right) \right] \right\}.$$

It is now possible to use the Ricci stability arguments to fix α and β. Ricci stability requires $f'(R) > 0$ and

$f''(R) \geq 0$ in order to have reliable late times cosmologies. The second derivative can be related to the absence of ghosts in the quantum theory[12]. The first derivative is given by:

$$f'(R) = 1 - R^\beta q(\beta+1)\alpha + q\epsilon(\beta+1)R^\beta \log\left(\frac{a_o^\beta R^\beta}{c}\right). \quad (24)$$

It is easy to see that this quantity is always positive if $\alpha \geq 0$ and $0 \leq \epsilon \leq \frac{e}{4}(1 + \frac{4}{e}\alpha)$. The second derivative is given by:

$$f''(R) = \beta(\beta + 1)\epsilon q R^{\beta - 1}. \qquad (25)$$

We see the special role played by the $\beta = 1$ case, in which it is independent from the curvature and depends only on the q and the ϵ parameters. This quantity is also always positive for positive curvature (which here we are forced to consider for non even β's). We also see that the requirement of having a positive β enters here. Note that the current understanding is that the ϵ parameter seems to be $3/2$. This constrains α to be bigger than $3/2 - e/4 \approx 0.82$.

V. CONCLUSIONS

In this paper we introduced a method to recover from the entropy of black holes an effective action. We obtained an effective action from the black holes entropy in LQG and in a general class of theory which originates from a logarithmic correction to the entropy. The langrangians follow (apart from the case of the Hawking entropy) from the a relation between the Ricci scalar and event horizon area and assuming that the theory is a of the form $f(R)$. This weakness however could be cured within each theory of quantum gravity if such relation is known for only a particular case.

In the case of LQG, even though the dynamics of the theory is not well understood yet, this procedure gave independently an action that in the limit of the Planck length going to zero reduces to the Einstein-Hilbert one. Further work should be done in order to understand if such procedure could turn out to be useful or not. In general we expect to consider a more general action which contains also other terms, such as

$$S = \int d^d x \sqrt{-g} \big[R + \alpha_1 R^2 + \alpha_2 R_{\alpha\beta} R^{\alpha\beta}$$
$$+ \alpha_3 R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + \alpha_4 R_{\alpha\beta\mu\nu} R^{\alpha\beta\rho\sigma} R^{\mu\nu}_{\rho\sigma} \dots \big]. \quad (26)$$

For this type of action formulas (5) and (7) will fix the free parameters α_i provided we know the relation between the tensors in (26) evaluated on the horizon and the event horizon area. In particular e want to stress the similarity between the class of Lagrangians obtained in this paper and the one obtained in [18] in the context of the Loop Quantum Cosmology.

The fact that each intertwiner may contain information on the action is intriguing. Mainly we treated all the intertwiners as tiny black hole with an entropy associated to their isolated horizons and assumed their entropy as the starting point to use the procedure we described in this paper. The idea of a quantum foam of black holes was first introduced by Smolin and Crane before the advent of LQG. In [19] it was assumed a distribution of black holes at the Planck scale and they have been used as a natural regulator. Moreover recently there has been a growing interest in the fractal structure of quantum theories of spacetime. 't Hooft in [20] argued with general arguments that a field close to the surface of a black hole experiences a dimensional reduction (for a recent paper on the subject see [21]). Carlip [22] pointed out that this seems to be a feature of several approaches to quantum gravity [23][24]. Being *naïve*, we are pushed to the idea of a deeper structure[25] of spacetime, that in the case of loop quantum gravity may rely on the intertwiner degree of freedom, but that in an ultimate theory of spacetime may be something more, as Jacobson and Padmanabhan have underlined.

[1] J Lee, R M Wald, Local symmetries and constraints, J. Math. Phys., vol. **31**, No. 3; R M Wald, Black Hole Entropy is Noether Charge, Phys. Rev. **D48** 3427-3431 (1993); V Iyer, R M Wald, Some Properties of Noether Charge and a Proposal for Dynamical Black Hole Entropy, Phys. Rev. **D50** (1994) 846-864

[2] T. Padmanabhan, Gravitational entropy of static spacetimes and microscopic density of states, (IUCAA, Pune) Class. Quant. Grav. 21, 4485-4494 (2004) [e-Print: gr-qc/0308070]; A Mukhopadhyay, T Padmanabhan, Holography of Gravitational Action Functionals Phys. Rev. **D 74**, 124023 (2006) [hep-th/0608120]

[3] T Padmanabhan, Thermodynamical aspects of gravity: new insights [arXiv:0911.5004v1]

[4] T. Padmanabhan, Gravity: A New Holographic Perspective, Int. J. Mod. Phys. D15 (2006) 1659-1676 [gr-qc/0606061]; T. Padmanabhan, A new perspective on Gravity and the dynamics of Spacetime, Int. J. Mod. Phys. D14 (2005) 2263-2270 [gr-qc/0510015v2]; T.Padmanabhan, The Holography of Gravity encoded in a relation between Entropy, Horizon area and Action for gravity, Gen.Rel.Grav. 34 (2002) 2029-2035 [gr-qc/0205090v1]

[5] T Jacobson, Thermodynamics of spacetime: the Einstein equation of state, Phys.Rev.Lett. 75 (1995) 1260-1263, [arXiv:gr-qc/9504004]

[6] Erik P Verlinde, On the Origin of Gravity and the Laws of Newton [arXiv:1001.0785v1]; J Makela, Quantum-Mechanical Model of Spacetime, [gr-qc/0701128];

[7] L Smolin, Newtonian gravity in loop quantum gravity, [arXiv:1001.3668]

[8] R C Myers, Black Holes in higher curvature gravity, Black Holes, Gravitational Radiation, and the Universe: Essays in Honor of C.V. Vishveshwara, eds., C.V. Vishveshwara, B.R. Iyer, B. Bhawal (1998) and references therein

[9] R Brustein, D Gorbonos, M Hadad, Wald's entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling, Phys. Rev. D 79, 044025 (2009) [arXiv:0712.3206v3]

[10] Rong-Gen Cai, Sang Pyo Kim, First law of thermodynamics and Friedmann equations of Friedmann-Robertson-Walker universe, JHEP 0502:050 (2005) [e-Print: hep-th/0501055]; R A Konoplya, Entropic force, holography and thermodynamics for static space-times, [arXiv:1002.2818v2 [hep-th]]

[11] L Modesto, Loop quantum black hole, Class. Quant. Grav. 23, 5587-5602 (2006) [e-Print: gr-qc/0509078]; L Modesto, Space-Time Structure of Loop Quantum Black Hole, [arXiv:0811.2196 [gr-qc]]; L Modesto I Premont-Schwarz, Self-dual Black Holes in LQG: Theory and Phenomenology, Phys. Rev. D80, 06404 (2009) [arXiv:0905.3170]

[12] V Faraoni, T P Sotiriou, $f(R)$ Theories Of Gravity, [arXiv:0805.1726]

[13] E Elizalde, P J Silva, F(R) gravity equationi of state, [arXiv:0804.3721]

[14] K Krasnov, C Rovelli, Black holes in full quantum gravity, Class. Quant. Grav. 26, 245009 (2009) [arXiv:0905.4916 [gr-qc]]; J Engle, A Perez, K Noui, Black hole entropy and SU(2) Chern-Simons theory, [arXiv:0905.3168]; C Rovelli, Black Hole Entropy from Loop Quantum Gravity, Phys. Rev. Lett.77:3288-3291 (1996), [arXiv:gr-qc/9603063]; O Dreyer, F Markopoulou, L Smolin, Symmetry and entropy of black hole horizons, Nucl.Phys. B744 (2006) [arXiv:hep-th/0409056]

[15] A Strominger, C Vafa, Microscopic origin of the Bekenstein-Hawking entropy, Phys.Lett.B379:99-104,1996, [arXiv:9601029v2[hep-th]]

[16] Rong-Gen Cai, Li-Ming Cao, Nobuyoshi Ohta, Black Holes in Gravity with Conformal Anomaly and Logarithmic Term in Black Hole Entropy, [e-Print: arXiv:0911.4379 [hep-th]]

[17] E Bianchi, Black Hole Entropy, Loop Gravity, and Polymer Physics, [arXiv:1011.5628]

[18] P Singh, Covariant Effective Action for Loop Quantum Cosmology a la Palatini, JCAP 0901, 030 (2009) [arXiv:0806.2783v2]

[19] J A Wheeler, K Ford. Geons, Black Holes, and Quantum Foam, 1995; L Smolin, L Crane, Space-Time Foam as the Universal Regulator, General Relativity and Gravitation, Vol. 17, No. 12 (1985)

[20] Dimensional Reduction in Quantum Gravity, G 't Hooft [arXiv:gr-qc/9310026v2];

[21] Vanishing Dimensions and Planar Events at the LHC, L Anchordoqui, De Chang D, M Fairbairn, G Landsberg, D Stojkovic, Nature 466 (2010) 426, [arXiv:1003.5914]

[22] Spontaneous Dimensional Reduction in Short-Distance Quantum Gravity? S Carlip, Proceedings of the XXV Max Born Symposium, Wroclaw (Poland) [arXiv:0909.3329v1]

[23] J Ambjorn, J Jurkiewicz, , R Loll, Phys. Rev. Lett. 95, 171301 (2005) [arXiv:hep-th/0505113]; O Lauscher, M Reuter, JHEP 0510 050 (2005) [arXiv:hep-th/0508202]; L Modesto, Fractal Structure of Loop Quantum Gravity, [arXiv:0812.2214[gr-qc]]; P Horava, Phys. Rev. D79, 084008 (2009) [arXiv:0901.3775[hep-th]]

[24] L Modesto, P Nicolini, The microstructure of a quantum universe [arXiv:0912.0220 [hep-ph]]; P Nicolini, Int. J. Mod. Phys. A24, 1229-1308 (2009) [arXiv:0807.1939 [hep-th]]; Noncommutative Black Holes, The Final Appeal To Quantum Gravity: A Review P Nicolini, A Smailagic and E Spallucci, Noncommutative geometry inspired Schwarzschild black hole, Phys. Lett. B 632, 547 (2006) [arXiv:gr-qc/0510112]

[25] T Padmanabhan, Entropy of Static Spacetimes and Microscopic Density of States,Class.Quant.Grav. 21 (2004) 4485-4494, [arXiv:0308070[gr-qc]]